Approximate Solution of Multipoint Boundary Value Problems

A.A. Opanuga, O.O. Agboola and H.I. Okagbue
Department of Mathematical Sciences, Covenant University, Canaanland, Ota, Nigeria

Abstract: This study applies the Differential Transform Method (DTM) to obtain the approximate solution of multipoint boundary value problems. Two examples are solved to illustrate the efficiency of the method. Comparison with the solution obtained by Adomian Decomposition Method revealed that the DTM is an excellent method for this type of problems.

Key words: Multipoint boundary value problems, Differential Transform Method, Adomian Decomposition Method, series solution, efficiency

INTRODUCTION

Boundary value problems occur in many areas such as engineering, technology control, optimization, etc. Specifically, in engineering it occurs in optimal control, beam deflections, fluid dynamics, hydrodynamics and hydro magnetics stability. Numerical analysis of boundary value problems has been studied by researchers like Wazwaz (2001), He (2003) and Adesanya et al. (2013).

Multipoint boundary value problems are boundary value problems with greater boundary conditions than the order of the differential equation. This type of differential equation has received greater attention in the recent years owing to its wide applications. For example, the vibrations in a wire of uniform cross sections and composed of materials with different densities, theory of elasticity and fluid flow through porous plate. Much research has been carried out in the area of the theoretical analysis of second order (Liang et al., 2008, Li and Shen, 2006) and third order multipoint boundary value problems (Sun, 2005; Shi et al., 2014; Yang, 2008). Few numerical analysis of multipoint boundary value problems are available in literature these include: Tatari and Dehghan (2006), Das et al. (2010), Liu and Wv (2002) and Akram et al. (2013). Differential Transform Method (DTM) is applied in this work and the results are compared with numerical solution by Adomian Decomposition Method (ADM). The DTM was first introduced by Zhou. This method has been adopted by numerous researchers like Edeki et al. (2014, 2015) to various physical situations because of its accuracy, simplicity and rapid convergence to the exact solution.

MATERIALS AND METHODS

Analysis of Differential Transform and Adomian Decomposition Methods

Fundamentals of Differential Transform Method: Let the arbitrary function \( y = u(t) \) be expressed in Taylor series about a point \( t = t_0 \) as:

\[
 u(t) = \sum_{k=0}^{\infty} \frac{d^k u}{dt^k} \bigg|_{t=t_0} (t-t_0)^k
\]

with the differential transformation of \( u(k) \) given as:

\[
 U(k) = \frac{1}{k!} \frac{d^k u}{dt^k} \bigg|_{t=t_0}
\]

We then obtain the inverse differential transform of \( U(k) \) as:

\[
 u(t) = \sum_{k=0}^{\infty} t^k Y(k)
\]

The following theorems can be obtained from Eq. 1-3 (Table 1):

(i) If \( u(t) = v_i(t) \), then \( U(k) = V_i(k) \)

(ii) If \( \frac{d^{m} u(t)}{dt^{m}} = v_i(t) \), then \( \frac{d^{m} U(k)}{dt^{m}} = V_i(k) \)

(iii) If \( u(t) = \frac{d^{m} v_i(t)}{dt^{m}} \), then \( U(k) = \frac{(k-m)!}{k!} V_i(k+m) \)

(iv) If \( u(t) = v_i(t) \), then \( U(k) = \sum_{n=0}^{\infty} V_i(n) V_i(k-n) \)

(v) If \( u(t) = t^k \), then \( U(k) = k! \delta(k-b) \) where

\[
 \delta(k-b) = \begin{cases} 0, & \text{if } k \neq b \\ 1, & \text{if } k = b \end{cases}
\]

(vi) If \( u(t) = v_i(t) \), then \( U(k) = \sum_{n=0}^{\infty} (k-n)! V_i(n) V_i(k+n) \)

Corresponding Author: A.A. Opanuga, Department of Mathematical Sciences, Covenant University, Canaanland, Ota, Nigeria
Fundamentals of Adomian Decomposition Method:
Consider the differential equation:
\[ L y + R y + N y = g \]  
(4)

Where:
\( L \) = Third order operator
\( R \) = Linear differential operator
\( Ny \) = The non-linear term
\( g \) = The source term

\( L \) is a fourth order operator and it is assumed to be invertible. Applying its inverse operator \( L^{-1} \) to both sides of Eq. 4 and using the boundary conditions, we obtain:

\[ y(t) = y(0) + ty'(0) + \frac{t^2}{2!}y''(0) + \frac{t^3}{3!}y'''(0) + L^{-1}(g) - L^{-1}(Ry) - L^{-1}(Ny) \]  
(5)

Then, \( y(t) \) can be written as:

\[ y(t) = \sum_{n=0}^{\infty} y_n \]  
(6)

The \( n \)th component \( y_n(t) \) include all the terms from the boundary conditions and the source term \( g \), i.e.:

\[ y_n(t) = y(0) + ty'(0) + \frac{A t^2}{2!} + \frac{B t^3}{3!} + L^{-1}(g) \]  
(7)

with constants:
\[ A = y'(0), \quad B = y''(0) \]  
(8)

which will be determined at \( t = 1 \). The remaining terms are embedded in the recursive relation:

\[ y_{n+1}(t) = L^{-1}(Ry_n) - L^{-1}(Ny_n), \quad n \geq 0 \]  
(9)

The non-linear term is determined by an infinite series of Adomian polynomials which can be calculated using:

\[ A_n = \frac{1}{n!} \frac{d^n}{dt^n} \left( \sum_{n=0}^{\infty} t^n y_n \right) \]  
(10)

RESULTS AND DISCUSSION

Examples: The methods are applied to the two examples with multipoint boundary conditions.

Example 1: Consider the third order differential equation that shows the shear deformation of sandwich beams:

\[ y'''(t) - k^2 y(t) - a = 0 \]  
(11)

with the boundary conditions:

\[ y'(0) = y'(1) = y(0.5) = 0 \]  
(12)

The constants \( k = 5 \) and \( a = 1 \). The theoretical solution of the equation is given as:

\[ y(t) = \frac{a}{k^2} \left( \sinh \frac{k}{2} t - \sinh kt \right) + \frac{a}{k^2} \left( t - \frac{1}{2} \right) + \]  
(13)

or

\[ \frac{a}{k^2} \tanh \frac{k}{2} \left( \cosh kt - \cosh \frac{k}{2} \right) \]

Solution by Differential Transform Method:
Transforming Eq. 11 with the boundary conditions Eq. 12, we obtain:

\[ Y(k + 3) = \frac{1}{(k + 3)!} (25k + 1)(\delta(k)) \]  
(14)

\[ Y(0) = A, \quad Y(1) = 0, \quad Y(2) = B \]  
(15)

Substituting Eq. 15 in Eq. 14, yields the following:

\[ Y(3) = \frac{1}{6}, \quad Y(4) = \frac{25B}{24}, \quad Y(5) = \frac{5}{24}, \quad Y(6) = \frac{125B}{144}, \quad Y(7) = \frac{125}{1008}, \quad Y(8) = \frac{3125B}{8064}, \quad Y(9) = \frac{3125}{72576}, \quad Y(10) = \frac{15625B}{145152}, \cdots \]
Collecting the terms together, we have:

\[ y(t) = A + B t + \frac{1}{2} t^3 + \frac{25B}{24} t^4 + \frac{5B}{24} t^5 + \frac{125B}{144} t^6 + \ldots \quad (16) \]

We now impose the boundary conditions at \( t = 1 \) and \( t = 0.5 \) using Table 1 to obtain the following system of equations:

\[ -0.02840163002 + A + \frac{0.2052914037B}{56048557} = 0 \]
\[ \frac{2624983B}{1916064} - \frac{2624983B}{177408} = 0 \quad (17) \]

Solving the system of equations above yields the constants \( A \) and \( B \):

\[ A = -0.01210708562, \quad B = 0.1973228596 \quad (18) \]

Using Eq. 17 in Eq. 16 gives the series solution as:

\[ y(t) = -0.012107085625 + 0.09866142980t^2 - \frac{1}{6} t^4 + \frac{0.20554464654t^5}{24} + \frac{0.1712872045t^6}{1008} \quad (19) \]

**Solution by Adomian Decomposition Method:**

Applying \( L^{-1} \) on Eq. 11 and imposing the boundary conditions Eq. 12 at \( t = 0 \), we obtain:

\[ y(t) = y(0) + ty'(0) + \frac{t^2}{2} y''(0) - L^{-1}(1) + 25L^{-1}(y'(0)) \quad (20) \]

the constants \( A - y(0) \) and \( B - y'(0) \) are to be determined later. The 0th component is identified as:

\[ y_0(t) = A + \frac{Bt^2}{2} - L^{-1}(1) = A + \frac{Bt^2}{2} \quad (21) \]

and recursive relation is given as:

\[ y_{n+1}(t) = 25L^{-1}(y'(n)) = 25L^{-1}(A_n) \quad (22) \]

at \( n = 0, 1, 2, \ldots \), we obtain:

\[ y_1 = 25L^{-1}(y'(0)) = \frac{25B}{24} t^4 - \frac{5B}{24} t^5 \]
\[ y_2 = 25L^{-1}(y'(1)) = \frac{125B}{144} t^6 - \frac{125B}{1008} t^7 \quad (23) \]

Collecting all the terms together, we have:

\[ y(t) = A + \frac{B}{2} t^2 + \frac{25B}{24} t^4 + \frac{5B}{24} t^5 + \frac{125B}{144} t^6 + \frac{125B}{1008} t^7 + \ldots \quad (24) \]

To evaluate the constants \( A \) and \( B \), we apply the boundary conditions at \( t = 1 \) for \( y(t) \) and \( t = 0.5 \) for \( y''(t) \) which yield the system of Eq. 25:

\[ 14.84064211556B + A = 2.92839794099 \]
\[ 0.2052915793 + A = 0.02840163584 \quad (25) \]

Solving the system of equations gives (Table 2 and Fig. 1):

\[ A = -0.01210708562, \quad B = 0.1973228596 \quad (26) \]

We obtain the series solution by substituting Eq. 26 in Eq. 24, giving:

\[ y(t) = -0.012107085625 + 0.09866142980t^2 - \frac{1}{6} t^4 + \frac{0.20554464654t^5}{24} + \frac{0.1712872045t^6}{1008} + \ldots \quad (27) \]

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<th>ADM solution</th>
<th>DTM error</th>
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**Fig. 1:** Plot of DTM and ADM solution of example 1
Example 2: We now consider a four point non-linear differential equation:

\[ y'''(t) = 4t^4 + 24y - y' \]  
\[ (28) \]

The boundary conditions are:

\[ y(0) = 0, y(1) = 1, y'(0.5) = 3, y'''(0.25) = 6 \]  
\[ (29) \]

and the theoretical solution of Eq. 28 is given as:

\[ y(t) = t^4 \]  
\[ (30) \]

Solution by Adomian Decomposition Method:

Applying \( L^{-1} \) on Eq. 28 and imposing the boundary conditions Eq. 29 at \( x = 0 \), we obtain:

\[ y(t) = y(0) + ty'(0) + \frac{t^2}{2!} y''(0) + \frac{t^3}{3!} y'''(0) - L^{-1}(4t^4) + L^{-1}(24) \]  
\[ (36) \]

The 0th component is written as:

\[ y_0(t) = Bt + \frac{C}{2!} t^2 + D \frac{t^3}{3!} + \frac{t^4}{1980} + t^4 \]  
\[ (37) \]

And the recursive relation is of the form:

\[ y_{n+1}(t) = 25L^{-1}(yy') = 25L^{-1}(A_n) \]  
\[ (38) \]

the Adomian polynomials for the non-linear part of the form \( yy' \) is written as follows:

\[ A_0 = y_0 y'_0 \]
\[ A_1 = y_1 y'_1 + y'_0 y_0 \]
\[ A_2 = y_2 y'_2 + y_1 y'_1 + y_0 y'_0 \]
\[ \vdots \]
\[ (39) \]

Using Eq. 39 in Eq. 38, we obtain the following iterates:

\[ y_1 = 25L^{-1}(yy'_0) \]
\[ y_2 = 25L^{-1}(yy'_1 + y'_0 y_0) \]  
\[ (40) \]

In this example, only three terms are used because of the large computations involved. More iterates will enhance the convergence. To evaluate constants \( B \), \( C \) and \( D \), we apply the boundary conditions Eq. 29 at \( t = 1, t = 0.5 \) for \( y''(t) \) and \( t = 0.5 \) for \( y'''(t) \) and solving the resulting system of equations, we obtain (Table 3 and Fig. 2):

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<th>DTM solution</th>
<th>ADM error</th>
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The series solution is written:

\[ y(t) = -1.674841180 \times 10^{-14}t - 1.772806171 \times 10^{-14}t^3 - 2.726727080 \times 10^{-13}t^4 + 2.337577482 \times 10^{-14}t^5 + \ldots \]  

(42)

CONCLUSION

In this research, Differential Transform Method has been applied to obtain the numerical solution of multipoint boundary value problems. Comparison was made with the results obtained by Adomian Decomposition Method. The two examples discussed the revealed the simplicity and effectiveness of DTM to handle various problems after the transformation of the differential equation, unlike ADM that requires the calculation of Adomian polynomials.

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REFERENCES


